Analele Universităţii Bucureşti, Matematică
Anul XLIX (2000), nr. 1, pp. 53–62

Equivalent definitions for Lipschitz compact connected manifolds

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(Received Mars 23, 1998)

Abstract - In this paper we present an alternative definition for Lipschitz manifolds, modelled in a real normed space.

Key words and phrases: Lipschitz manifolds
Mathematics Subject Classification: 58B99, 57N99

1 Introduction

The purpose of this paper is to show that for a compact connected Lipschitz manifold modelled on a real normed space, the definition based on atlases is essentialy the same with the following: such a manifold is a metric space that locally is Lipschitz equivalent to an open set from the modelling space.

The initial motivation for our work comes from [2] where the corresponding result for LIP-n manifolds is given.

In the following we will be more precisely. First let us recall some basic facts.

Definition 1 *If (X, d) and (Y, d') are metric spaces, a map $f : X \to Y$ is said to be Lipschitz, if there is a constant $M \geq 0$ such that $d'(f(x), f(y)) \leq M \cdot d(x, y)$ for all x, y in X and $lip(f)$ is defined as the least such a constant. If every $x \in X$ has a neighborhood U such that $f_{|U}$ is Lipschitz, f is said to be locally Lipschitz (abbreviated LIP).*

Remark 1 *Intuitively speaking, a Lipschitz (LIP) map is one that obeys (temporary) speed limits.*

Theorem 1 (Rademacher, see [3]) . *If U is an open set in $\mathbf{R}^n$ and $f : U \to \mathbf{R}^m$ is a Lipschitz map , then f is differentiable outside of a Lebesgue null subset of U.*

Now we define the Lipschitz (LIP) manifolds.

DEFINITION 2 (see [5], p. 42) *A compact, oriented, n-topological manifold M, without boundary, is a Lipschitz manifold, if there is a family $(U_i, h_i)_{i\in I}$, where $(U_i)_{i\in I}$ is an open cover of M, $h_i : U_i \to V_i \subseteq \mathbf{R}^n$ is a homeomorphism from U_i onto an open subset V_i of $\mathbf{R}^n$ and $h_i \circ h_j^{-1} : h_j(U_i \cap U_j) \to h_i(U_i \cap U_j)$ are Lipschitz for all $i, j \in I, U_i \cap U_j \neq \emptyset$.*

DEFINITION 3 (see [2], p. 97) *A LIP n-manifold is a Hausdorff topological space M, such that there is a family $(U_i, h_i)_{i\in I}$, where $(U_i)_{i\in I}$ is an open cover of M, $h_i : U_i \to U_i'$ is a homeomorphism, U_i' being open either in $\mathbf{R}^n$ or $\mathbf{R}^n_+$, and $h_i \circ h_j^{-1} : h_j(U_i \cap U_j) \to h_i(U_i \cap U_j)$ are LIP for all $i, j \in I, U_i \cap U_j \neq \emptyset$.*

The key features of a Lipschitz (LIP) manifold are that, on one hand it seems to be only slightly weaker than a smooth structure, so that one can still do analysis with it (see [5]), and yet essential uniqueness of this structure is almost automatic in many situations, that are very far from being smooth (see [4]).

In [2] it is proved, using strong results, like an embedding theorem or a special case of a metrization theorem for locally metric spaces, that, for second countable spaces, Definition 3 is essentially the same with:

DEFINITION 4 (see [2], p. 97) *A LIP n-manifold is a separable metric space M, such that every point $x \in M$ has a closed neighborhood U_x for which there is a bijection $f_x : U_x \to [-1, 1]^n$, such that f_x and f_x^{-1} are LIP.*

In this paper we give, by a direct proof, using no other results, a similar alternative definition for compact connected Lipschitz manifolds, modelled on a real normed space (see theorem below).

2 The result

THEOREM 2 *Let $\mathcal{X}$ be a real normed space. Then the following statements are equivalent:*

a) M is a compact, connected, topological space, for which there exists a family $(U_j, h_j)_{j\in\{1,...,n\}}$, $n \in \mathbf{N}^*$, where $(U_j)_{j\in\{1,...,n\}}$ is an open cover of M and $h_j : W_j = \overset{\circ}{W_j} \subseteq \mathcal{X} \to U_j$ is a homeomorphism for all $j \in \{1, ..., n\}$, such that $h_i^{-1} \circ h_j : h_j^{-1}(U_i \cap U_j) \to h_i^{-1}(U_i \cap U_j)$ is Lipschitz for all $i, j \in \{1, ..., n\}$, so that $U_i \cap U_j \neq \emptyset$.

b) M is a compact, connected, metric space, for which there exists a family $(V_p, g_p)_{p\in\{1,...,m\}}$, $m \in \mathbf{N}^*$, where $(V_p)_{p\in\{1,...,m\}}$ is an open cover of M and $g_p : W_p' = \overset{\circ}{W_p'} \subseteq \mathcal{X} \to V_p$ is a bijection, such that g_p and g_p^{-1} are Lipschitz for all $p \in \{1,...,m\}$.

Let us observe that the affirmation a) is analogous to the Definition 2, for Lipschitz manifolds modelled in a real normed space.
Proof. Obviously $b) \Rightarrow a)$. For $a) \Rightarrow b)$, we first construct

- **d a pseudo-metric on M.**

Let us consider a partition of unity $(f_j)_{j\in\{1,...,n\}}$ related to the open cover $(U_j)_{j\in\{1,...,n\}}$ of M. In the following we will use the notation

$$J(x,y) = \{j \in \{1,...,n\} \mid x, y \in f_j^{-1}((0,1])\}.$$

We define $d : M \times M \to \mathbf{R}$ by

$$\begin{aligned} d(x,y) \;=\; \inf\Big\{ &\sum_{i=1}^{t} \frac{1}{\sum\limits_{j\in J(x_{i-1},x_i)} f_j(x_{i-1})\cdot f_j(x_i)} \cdot \sum_{j\in J(x_{i-1},x_i)} f_j(x_{i-1}) \times \\ & f_j(x_i)\cdot \left\| h_j^{-1}(x_{i-1}) - h_j^{-1}(x_i)\right\| \mid t \in \mathbf{N}^*\ ,\ x = x_0, ..., x_{t-1}, \\ & x_t = y \in M \text{ so that } J(x_{i-1}, x_i) \neq \emptyset \text{ for all } i \in \{1,...,t\}\Big\}. \end{aligned}$$

It is easy to see that d is a pseudometric on M.

- **d gives the original topology on M.**

First we prove that

$$Id : (M,d) \to M \text{ is continuous} \tag{1}$$

Indeed, if $x_0 \in M$ there is $j_0' \in \{1,...,n\}$ so that $x_0 \in f_{j_0'}^{-1}((0,1]) \subseteq U_{j_0'}$. Let us take $W \in \mathcal{V}_{x_0}$. We can consider $V \in \mathcal{V}_{x_0}$ such that $V \subseteq f_{j_0'}^{-1}((0,1]) \subseteq U_{j_0'}$, $V \subseteq W$ and for all $j \in \{1,...,n\}$ such that $U_j \cap V \neq \emptyset$, we have:

$$V \subseteq U_j \text{ or } V \cap f_j^{-1}((0,1]) = \emptyset.$$

We can construct such a V in the following way: If $x_0 \in U_j$ we consider $V_j = W \cap f_{j_0'}^{-1}((0,1]) \cap U_j$. If $x_0 \notin U_j$, as $\overline{f_j^{-1}((0,1])} \subseteq U_j$ we have $x_0 \notin \overline{f_j^{-1}((0,1])}$; since X is normal (being compact), there is $U^j \in \mathcal{V}_{x_0}$ so that $U^j \cap \overline{f_j^{-1}((0,1])} = \emptyset$; then we consider $V_j = W \cap f_{j_0'}^{-1}((0,1]) \cap U^j$. Finally we take $V = \bigcap_{j\in\{1,...,n\}} V_j$. As h_j, for $j \in \{1,...,n\}$, is homeomorphism, we can consider $r > 0$ so that for all $j \in \{1,...,n\}$, with the property that $V \subseteq U_j$, we have

$$h_j(B_r(h_j^{-1}(x_0))) = \{x \in M \mid \left\|h_j^{-1}(x) - h_j^{-1}(x_0)\right\| < r\} \subseteq V.$$

Now we show that

$$B_{\frac{r}{m}}(x_0) \subseteq V \subseteq W, \tag{2}$$

where $m = \max_{\substack{i,j \in \{1,...,n\} \\ U_i \cap U_j \neq \emptyset}} lip(h_j^{-1} o h_i)$, so we deduce that $Id : (M, d) \to M$ is continuous.

In order to prove the inclusion above, let us take $y \in B_{\frac{r}{m}}(x_0)$, i.e. $d(x_0, y) < \frac{r}{m}$. Therefore, there are $t \in \mathbb{N}^*$, $x = x_0, ..., x_{t-1}, x_t = y \in M$, so that $J(x_{i-1}, x_i) \neq \emptyset$ for all $i \in \{1,...,t\}$, such that

$$\sum_{i=1} \frac{1}{\sum\limits_{j\in J(x_{i-1},x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \times \sum_{j\in J(x_{i-1},x_i)} f_j(x_{i-1}) \cdot f_j(x_i) \cdot \left\|h_j^{-1}(x_{i-1}) - h_j^{-1}(x_i)\right\| < \frac{r}{m}.$$

Now we show that, if $x_i \in V$ for all $i \in \{0,...,q\}$, where $q \in \{0,...,t-1\}$, then $x_{q+1} \in V$. As $x_0 \in V$, then we obtain, by the assertion above, that $x_t = y \in V$, so (2) is true.

Let us prove the affirmation above. For all $j \in J(x_q, x_{q+1})$, $x_q \in V \cap f_j^{-1}((0,1])$, so, because of the properties of V, $V \subseteq U_j$. Hence x_0, x_1,..., $x_q \in V \subseteq U_j$ for all $j \in J(x_q, x_{q+1})$. If $j_0 \in J(x_q, x_{q+1})$, then $x_{q+1} \in f_{j_0}^{-1}((0,1]) \subseteq U_{j_0}$, and we obtain

$$\left\|h_{j_0}^{-1}(x_0) - h_{j_0}^{-1}(x_{q+1})\right\| \leq \sum_{i=1}^{q+1} \left\|h_{j_0}^{-1}(x_{i-1}) - h_{j_0}^{-1}(x_i)\right\| \leq$$

$$\leq \sum_{i=1}^{q+1} \frac{1}{\sum\limits_{j\in J(x_{i-1},x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \cdot \sum_{j\in J(x_{i-1},x_i)} f_j(x_{i-1}) \cdot f_j(x_i) \times$$

$$\left\|h_{j_0}^{-1}(x_{i-1}) - h_{j_0}^{-1}(x_i)\right\| = \sum_{i=1}^{q+1} \frac{1}{\sum\limits_{j\in J(x_{i-1},x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \times$$

$$\sum_{j\in J(x_{i-1},x_i)} f_j(x_{i-1})\cdot f_j(x_i)\cdot\left\|h_{j_0}^{-1}\circ h_j\circ h_j^{-1}(x_{i-1})-h_{j_0}^{-1}\circ h_j\circ h_j^{-1}(x_i)\right\|\le$$

$$\le m\sum_{i=1}^{t}\frac{1}{\sum\limits_{j\in J(x_{i-1},x_i)} f_j(x_{i-1})\cdot f_j(x_i)}\sum_{j\in J(x_{i-1},x_i)} f_j(x_{i-1})\cdot f_j(x_i)\times$$

$$\left\|h_j^{-1}(x_{i-1})-h_j^{-1}(x_i)\right\|<m\cdot\frac{r}{m}=r,$$

so, $x_{q+1}\in h_{j_0}(B_r(h_{j_0}^{-1}(x_0)))\subseteq V$

Now we prove that

$$Id: M\to(M,d)\text{ is continuous}\tag{3}$$

Indeed, if $x_0\in M$, then $x_0\in f_{j_0}^{-1}((0,1])$, ... , $f_{j_p}^{-1}((0,1])$ and $x_0\notin f_j^{-1}((0,1])$ for $j\in\{1,...,n\}\setminus\{j_0,...,j_p\}$. As $h_{j_l}^{-1}: f_{j_l}^{-1}((0,1])\to h_{j_l}^{-1}(f_{j_l}^{-1}((0,1]))$ is continuous in x_0 for all $l\in\{0,1,...,p\}$, we deduce that, for every $r>0$, there is $U'_{j_l}\in\mathcal{V}_{x_0}$, $U'_{j_l}\subseteq f_{j_l}^{-1}((0,1])$ such that

$$\left\|h_{j_l}^{-1}(y)-h_{j_l}^{-1}(x_0)\right\|<r,$$

for every $y\in U'_{j_l}$ and every $l\in\{0,1,...,p\}$. If $U=\bigcap\limits_{l=0}^{p} U'_{j_l}\in\mathcal{V}_{x_0}$, then $y\in U$ implies

$$d(x_0,y)\le\frac{1}{\sum\limits_{j\in J(x_0,y)} f_j(x_0)\cdot f_j(y)}\cdot\sum_{j\in J(x_0,y)} f_j(x_0)\,f_j(y)\cdot\left\|h_j^{-1}(y)-h_j^{-1}(x_0)\right\|\le$$

$$\le\max_{l\in\{0,1,...,p\}}\left\|h_{j_l}^{-1}(y)-h_{j_l}^{-1}(x_0)\right\|<r,$$

so, $U\subseteq B_r(x_0)$, i.e. $Id: M\to(M,d)$ is continuous. From (1) and (3) we infer that d gives the original topology on M.

• ***d* is a metric on *M*.**

Indeed, let us consider $x,y\in M$, so that $d(x,y)=0$. For $x'\in M$ arbitrary, since $\sum\limits_{j=1}^{n} f_j(x')=1$, there is $k_{x'}\in\{1,...,n\}$ such that $f_{k_{x'}}(x')\ne 0$, i.e. $x'\in f_{k_{x'}}^{-1}((0,1])$. As $f_{k_{x'}}^{-1}((0,1])$ is open, as d gives the original topology on M, we obtain:

for each $x'\in M$ there exist $k_{x'}\in\{1,...,n\}$ and $\delta_{x'}>0$, so that

$$B_{\delta_{x'}}(x')\subseteq B_{3\cdot\delta_{x'}}(x')\subseteq f_{k_{x'}}^{-1}((0,1])\subseteq U_{k_{x'}}.\tag{4}$$

If we consider $0 < \varepsilon < \delta_x$, then $0 = d(x,y) < \varepsilon < \delta_x$, so we can choose $t \in \mathbf{N}^*$, and $x = x_0, ..., x_{t-1}, x_t = y \in M$ so that $J(x_{i-1}, x_i) \neq \emptyset$ for all $i \in \{1, ..., t\}$, such that

$$\sum_{i=1}^{t} \frac{1}{\sum\limits_{j\in J(x_{i-1},x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \cdot \times$$
$$\sum_{j\in J(x_{i-1},x_i)} f_j(x_{i-1}) \cdot f_j(x_i) \cdot \left\| h_j^{-1}(x_{i-1}) - h_j^{-1}(x_i) \right\| < \varepsilon < \delta_x.$$

Then, for each $l, m \in \{0, ..., t\}$, $l < m$ we get

$$d(x_l, x_m) \leq \sum_{s=l+1}^{m} \frac{1}{\sum\limits_{j\in J(x_{s-1},x_s)} f_j(x_{s-1}) \cdot f_j(x_s)} \times$$
$$\sum_{j\in J(x_{i-1},x_i)} f_j(x_{s-1}) \cdot f_j(x_s) \cdot \left\| h_j^{-1}(x_{s-1}) - h_j^{-1}(x_s) \right\| \leq$$
$$\leq \sum_{i=1}^{t} \frac{1}{\sum\limits_{j\in J(x_{i-1},x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \cdot \sum_{j\in J(x_{i-1},x_i)} f_j(x_{i-1}) \cdot f_j(x_i) \times$$
$$\left\| h_j^{-1}(x_{i-1}) - h_j^{-1}(x_i) \right\| < \varepsilon < \delta_x,$$

so, $\{x_0, x_1, ..., x_t\} \subseteq B_{\delta_x}(x_0) = B_{\delta_{x_0}}(x_0)$. Using (4), we can choose $k_0 \in \{1, ..., n\}$ such that

$$\{x_0, x_1, ..., x_t\} \subseteq B_{\delta_{x_0}}(x_0) \subseteq B_{3\cdot\delta_{x_0}}(x_0) \subseteq f_{k_0}^{-1}((0,1]) \subseteq U_{k_0}.$$

Denoting $J = \{j \in \{1, ..., n\} \mid U_j \cap U_{k_0} \neq \emptyset\}$ and $m = \min\limits_{j\in J} \dfrac{1}{lip(h_{k_0}^{-1} \circ h_j)}$ we obtain

$$m \cdot \left\| h_{k_0}^{-1}(x) - h_{k_0}^{-1}(y) \right\| \leq m \cdot \sum_{i=1}^{t} \left\| h_{k_0}^{-1}(x_{i-1}) - h_{k_0}^{-1}(x_i) \right\| =$$
$$= m \cdot \sum_{i=1}^{t} \frac{1}{\sum\limits_{j\in J(x_{i-1},x_i)\cap J} f_j(x_{i-1}) \cdot f_j(x_i)} \cdot \sum_{j\in J(x_{i-1},x_i)\cap J} f_j(x_{i-1}) \cdot f_j(x_i) \times$$
$$\left\| h_{k_0}^{-1}(x_{i-1}) - h_{k_0}^{-1}(x_i) \right\| = m \cdot \sum_{i=1}^{t} \frac{1}{\sum\limits_{j\in J(x_{i-1},x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \times$$

$$\sum_{j\in J(x_{i-1},x_i)\cap J} f_j(x_{i-1})\cdot f_j(x_i)\cdot\left\|h_{k_0}^{-1}\circ h_j\circ h_j^{-1}(x_{i-1})-h_{k_0}^{-1}\circ h_j\circ h_j^{-1}(x_i)\right\|\leq$$

$$\leq\sum_{i=1}^{t}\frac{1}{\sum\limits_{j\in J(x_{i-1},x_i)} f_j(x_{i-1})\cdot f_j(x_i)}\cdot\sum_{j\in J(x_{i-1},x_i)} f_j(x_{i-1})\cdot f_j(x_i)\times$$

$$\left\|h_j^{-1}(x_{i-1})-h_j^{-1}(x_i)\right\|<\varepsilon,$$

for all ε, such that $0<\varepsilon<\delta_x$.

We infer that $h_{k_0}^{-1}(x)=h_{k_0}^{-1}(y)$, so, $x=y$, since h_{k_0} is homeomorphism.

• **The construction of the family** $(V_p,g_p)_{p\in\{1,\dots,m\}}$.

As M is is compact we can consider an open cover of M, $(V_p)_{p\in\{1,\dots,m\}}$, so that

$$V_p=B_{\frac{\delta_{x_p}}{2}}(x_p)\subseteq B_{3\cdot\delta_{x_p}}(x_p)\subseteq f_{k_{x_p}}^{-1}((0,1])\subseteq U_{k_{x_p}},\tag{5}$$

where $\delta_{x_p}>0$ and $k_{x_p}\in\{1,\dots,n\}$ are given by (4). We will denote in the following: $\delta_{x_p}=\delta_p$ and $k_{x_p}=k_p$. We consider the family $(V_p,g_p)_{p\in\{1,\dots,m\}}$, where

$$g_p^{-1}=h_{k_p}^{-1}|_{V_p}:V_p\subseteq U_{k_p}\to W_p'=g_p^{-1}(V_p)\subseteq\mathcal{X}.$$

• g_p **is Lipschitz for all** $p\in\{1,\dots,m\}$.

Indeed, for $X,Y\in W_p'=g_p^{-1}(V_p)=h_{k_p}^{-1}(V_p)$, because $h_{k_p}(X)$, $h_{k_p}(X)\in V_p\subseteq f_{k_p}^{-1}((0,1])$, we have

$$\begin{aligned}
d(g_p(X),g_p(Y)) &= d(h_{k_p}(X),h_{k_p}(X))\leq\\
&\leq \frac{1}{\sum\limits_{j\in J(h_{k_p}(X),h_{k_p}(Y))} f_j(h_{k_p}(X))\cdot f_j(h_{k_p}(Y))}\times\\
&\quad\sum_{j\in J(h_{k_p}(X),h_{k_p}(Y))} f_j(h_{k_p}(X))\cdot f_j(h_{k_p}(Y))\times\\
&\quad\left\|h_j^{-1}(h_{k_p}(X))-h_j^{-1}(h_{k_p}(Y))\right\|\leq\\
&\leq \max_{\substack{j\in\{1,\dots,n\}\\ U_j\cap U_{k_p}\neq\emptyset}} lip(h_j^{-1}\circ h_{k_p})\cdot\|X-Y\|
\end{aligned}$$

Hence, g_p is Lipschitz for all $p\in\{1,\dots,m\}$.

• g_p^{-1} **is Lipschitz for all** $p\in\{1,\dots,m\}$

Let us consider first $x,y\in V_p\subseteq f_{k_p}^{-1}((0,1])\subseteq U_{k_p}$, $t\in\mathbf{N}^*$ and $x=x_0,\dots,x_{t-1},x_t=y\in M$ so that $x_l\in f_{k_p}^{-1}((0,1])$ for each $l\in\{0,\dots,t\}$. If we denote $J_p = \{j \in \{1, ..., n\} \mid U_j \cap U_{k_p} \neq \emptyset\}$ and $m = \min\limits_{j \in J_p} \frac{1}{lip(h_{k_p}^{-1} \circ h_j)}$, we have

$$m \cdot \left\| g_p^{-1}(x) - g_p^{-1}(y) \right\| \leq m \cdot \sum_{i=1}^{t} \left\| h_{k_p}^{-1}(x_{i-1}) - h_{k_p}^{-1}(x_i) \right\| =$$

$$= m \cdot \sum_{i=1}^{t} \frac{1}{\sum\limits_{j \in J(x_{i-1}, x_i) \cap J_p} f_j(x_{i-1}) \cdot f_j(x_i)} \cdot \sum_{j \in J(x_{i-1}, x_i) \cap J_p} f_j(x_{i-1}) \cdot f_j(x_i) \times$$

$$\left\| h_{k_p}^{-1}(x_{i-1}) - h_{k_p}^{-1}(x_i) \right\| = m \cdot \sum_{i=1}^{t} \frac{1}{\sum\limits_{j \in J(x_{i-1}, x_i) \cap J_p} f_j(x_{i-1}) \cdot f_j(x_i)} \times$$

$$\sum_{j \in J(x_{i-1}, x_i) \cap J_p} f_j(x_{i-1}) \cdot f_j(x_i) \cdot \left\| h_{k_p}^{-1} \circ h_j \circ h_j^{-1}(x_{i-1}) - h_{k_p}^{-1} \circ h_j \circ h_j^{-1}(x_i) \right\| \leq$$

$$\leq \sum_{i=1}^{t} \frac{1}{\sum\limits_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \cdot \sum_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i) \times$$

$$\left\| h_j^{-1}(x_{i-1}) - h_j^{-1}(x_i) \right\|.$$

Now, if $x, y \in V_p$, $t \in \mathbf{N}^*$ and $x = x_0, ..., x_{t-1}, x_t = y \in M$, so that $J(x_{i-1}, x_i) \neq \emptyset$ for all $i \in \{1, ..., t\}$, there are two cases:

A :

$$\sum_{i=1}^{t} \frac{1}{\sum\limits_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \times$$
$$\sum_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i) \cdot \left\| h_j^{-1}(x_{i-1}) - h_j^{-1}(x_i) \right\| \leq \delta_p$$

and

B :

$$\sum_{i=1}^{t} \frac{1}{\sum\limits_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \times$$
$$\sum_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i) \cdot \left\| h_j^{-1}(x_{i-1}) - h_j^{-1}(x_i) \right\| > \delta_p.$$

For the case A, let us observe that for each $l \in \{1, ..., t\}$

$$d(x, x_l) \le \sum_{i=1}^{l} \frac{1}{\sum\limits_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \times \sum_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i) \cdot \left\| h_j^{-1}(x_{i-1}) - h_j^{-1}(x_i) \right\| \le$$

$$\le \sum_{i=1}^{t} \frac{1}{\sum\limits_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \times \sum_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i) \cdot \left\| h_j^{-1}(x_{i-1}) - h_j^{-1}(x_i) \right\| \le \delta_p.$$

Then, for each $l \in \{1, ..., t\}$

$$d(x_l, x_p) \le d(x_l, x) + d(x, x_p) < \delta_p + \delta_p = 2 \cdot \delta_p,$$

because $x \in V_p \subseteq B_{\delta_p}(x_p)$. Hence, according to (5), $x_l \in B_{2 \cdot \delta_p}(x_p) \subseteq B_{3 \cdot \delta_p}(x_p) \subseteq f_{k_p}^{-1}((0, 1])$, for all $l \in \{0, ..., t\}$. Taking into account the calculations made above, we have

$$\left\| g_p^{-1}(x) - g_p^{-1}(y) \right\| \le \frac{1}{m} \cdot \sum_{i=1}^{t} \frac{1}{\sum\limits_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i)} \times \sum_{j \in J(x_{i-1}, x_i)} f_j(x_{i-1}) \cdot f_j(x_i) \cdot \left\| h_j^{-1}(x_{i-1}) - h_j^{-1}(x_i) \right\| \tag{6}$$

For the case B, as $d(x, y) \le d(x, x_p) + d(x_p, y) < \delta_p$, there is $s \in N^*$ and $x = x'_0, ..., x'_{s-1}, x'_s = y \in M$ such that $J(x'_{i-1}, x'_i) \ne \emptyset$ for all $i \in \{1, ..., s\}$ and

$$\sum_{i=1}^{s} \frac{1}{\sum\limits_{j \in J(x'_{i-1}, x'_i)} f_j(x'_{i-1}) \cdot f_j(x'_i)} \times \sum_{j \in J(x_{i-1}, x_i)} f_j(x'_{i-1}) \cdot f_j(x'_i) \cdot \left\| h_j^{-1}(x'_{i-1}) - h_j^{-1}(x'_i) \right\| < \delta_p.$$

Then, according to the case A, we have

$$\left\| g_p^{-1}(x) - g_p^{-1}(y) \right\| \le \frac{1}{m} \cdot \sum_{i=1}^{s} \frac{1}{\sum\limits_{j \in J(x'_{i-1}, x'_i)} f_j(x'_{i-1}) \cdot f_j(x'_i)}$$

$$\sum_{j\in J(x_{i-1},x_i)} f_j(x_{i-1})\cdot f_j(x_i')\cdot\left\|h_j^{-1}(x_{i-1}')-h_j^{-1}(x_i')\right\|<$$

$$<\ \frac{\delta_p}{m}<\frac{1}{m}\cdot\sum_{i=1}^{t}\frac{1}{\sum\limits_{j\in J(x_{i-1},x_i)} f_j(x_{i-1})\cdot f_j(x_i)}\times$$

$$\sum_{j\in J(x_{i-1},x_i)} f_j(x_{i-1})\cdot f_j(x_i)\cdot\left\|h_j^{-1}(x_{i-1})-h_j^{-1}(x_i)\right\|$$

so, the inequality (6) is valid also for the case B.

Therefore, the inequality (6) implies that

$$\left\|g_p^{-1}(x)-g_p^{-1}(y)\right\|\leq\frac{1}{m}\cdot d(x,y)$$

for all $x,y\in V_p$, which permits us to conclude that g_p^{-1} is Lipschitz for all $p\in\{1,...,m\}$.

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